4/5/2021
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Before I forget how, let's build a blade-element model of a hovering helicopter.

Some basic parameters:

$D_{rotor} := 30\ ft$ rotor diameter $\quad R_{tip} := \frac{D_{rotor}}{2}$ $\quad \rho := 0.0765\ \frac{lb}{ft^3}$ air density

$M_{tip} := 0.5$ tip Mach Number (so we can calculate angular velocity)

$$\omega := \frac{M_{tip} \cdot 1000\ \frac{ft}{sec}}{R_{tip}} = 318.31\ rpm \qquad \text{check} \qquad \omega \cdot R_{tip} = 500\ \frac{ft}{s}$$

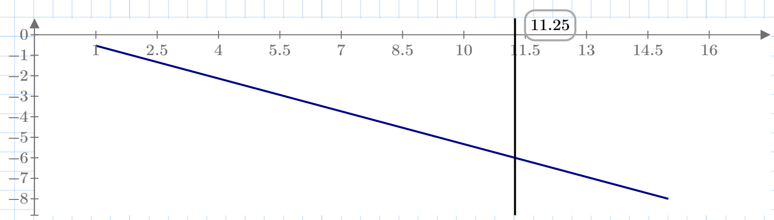
Let the blade be rectangular, the blade chord is $chd := 8\ in$. The rotor has four blades. Rotor ***solidity***, the ratio of blade(s) area to total disk area is $\sigma := \frac{4 \cdot chd}{\pi \cdot R_{tip}} = 0.057$. (Solidity will allow comparison with the performance of other rotors.

The airfoil of our blade will be a NACA 0012

NACA 0012 lift and drag

We want to twist the blade so that the inner sections are at a higher angle of attack than the outer ends. We'll at least start with a linear twist, with the outboard end eight degrees less than the inner. $\tau(r) := -8\ deg \cdot \frac{r}{R_{tip}}$

$rr := 1\ ft, 1.1\ ft .. R_{tip}$

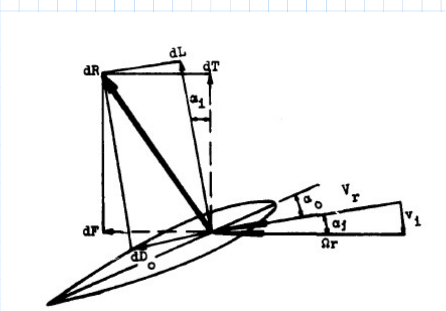


$\tau(rr)\ (deg)$

$rr\ (ft)$

Since we are only hovering we won't need to change the angles of the blades as they go around. Those are "cyclic" inputs. We only need to change the "collective" angle of the blades to change the rotor thrust. Define the collective angle as the angle of the blade at 75% radius. Then the ***physical*** local angle of attack is the twist combined with the collective: $\alpha_p(r, \theta_c) := \theta_c + \tau(r) - \tau(0.75\ R_{tip})$.

But the rotor producing thrust creates a downwash. This downwash changes the apparent angle of attack.



Let v_i be the induced velocity (normal to the plane of rotation.) Then $\alpha_i = \operatorname{atan}\left(\frac{v_i}{\omega \cdot r}\right)$, and $\alpha_o = \alpha_p - \alpha_i$ in this sketch.

$$vel(u, \omega, r) := \begin{bmatrix} \omega \cdot r \\ u \end{bmatrix} \qquad \alpha_i(vel) := \operatorname{atan}\left(\frac{vel_1}{vel_0}\right)$$

$$V_r(u, \omega, r) := \left| \begin{bmatrix} \omega \cdot r \\ u \end{bmatrix} \right| \qquad \alpha_o(r, u, \omega, \theta_c) := \alpha_p(r, \theta_c) - \operatorname{atan}\left(\frac{u}{\omega \cdot r}\right)$$

Here's the problem: At a given physical angle of attack the blade develops lift (and drag) which create downflow, which changes the apparent angle of attack, which reduces the downflow. ***How do we find the correct downflow where the thrust and downflow velocity match?***

$$dl(r,\theta_c,u) := C_L(\alpha_o(r,u,\omega,\theta_c)) \cdot \frac{chd \cdot \rho \cdot (\omega \cdot r)^2}{2}$$

$$Thrst(\theta_c,u) := 4 \cdot \int_{1\,ft}^{R_{tip}} dl(r,\theta_c,u)\,dr$$

Induced downflow

$$\frac{T}{A_D} = \Delta P = \frac{1}{2}\rho v_w^2 \qquad v_w = 2v_i$$

$$v_i(\theta_c) := \mathrm{root}\left(\frac{\sqrt{\dfrac{2 \cdot Thrst(\theta_c,u)}{\pi \cdot R_{tip}{}^2 \cdot \rho}}}{2} - u, u, 0\,\frac{ft}{sec}, 100\,\frac{ft}{sec}\right)$$

$$v_i(5\,deg) = 16.208\,\frac{ft}{sec}$$

$$ddl(r,\theta_c,u) := C_L(\alpha_o(r,u,\omega,\theta_c)) \cdot \frac{chd \cdot \rho \cdot (V_r(v_i(\theta_c),\omega,r))^2}{2}$$

$$Thrust(\theta_c) := 4 \cdot \int_{1\,ft}^{R_{tip}} ddl(r,\theta_c,v_i(\theta_c))\,dr$$

$$ddf(r,\theta_c,u) := \left(\frac{chd \cdot \rho \cdot (V_r(v_i(\theta_c),\omega,r))^2}{2}\right) \cdot \left(C_L(\alpha_o(r,u,\omega,\theta_c)) \cdot \sin\left(\mathrm{atan}\left(\frac{u}{\omega \cdot r}\right)\right) + C_D(\alpha_o(r,u,\omega,\theta_c)) \cdot \cos\left(\mathrm{atan}\left(\frac{u}{\omega \cdot r}\right)\right)\right)$$

$$Torq(\theta_c) := 4 \cdot \int_{1\,ft}^{R_{tip}} r \cdot ddf(r,\theta_c,v_i(\theta_c))\,dr$$

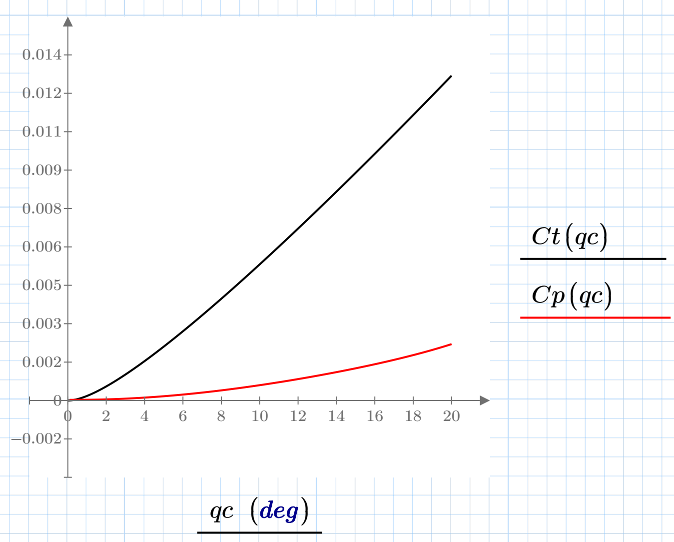
$$qc := 0\,deg, 0.1\,deg..20\,deg$$

$$\omega \cdot Torq(0) = 9.825\,hp$$

$$Ct(\theta_c) := \frac{Thrust(\theta_c)}{\rho \cdot \omega^2 \cdot \pi \cdot R_{tip}{}^4} \qquad Cq(\theta_c) := \frac{Torq(\theta_c)}{\rho \cdot \omega^2 \cdot \pi \cdot R_{tip}{}^5}$$

$$Cp(\theta_c) := \frac{\omega \cdot Torq(\theta_c) + \sqrt{\dfrac{Thrust(\theta_c)^3}{2 \cdot \rho \cdot \pi \cdot R_{tip}{}^2}}}{\rho \cdot \omega^3 \cdot \pi \cdot R_{tip}{}^5}$$

$$Thrust(0.003\,deg) = 0\,lbf$$



We usually define zero collective as the angle where thrust is zero. At a first guess, we came ~~pretty~~ close.

$$\sqrt{\frac{Thrust(1)^3}{2 \cdot \rho \cdot \pi \cdot R_{tip}{}^2}} = 2556.614\,hp$$

$$Cp(0) = 0$$

dF

The minimum p

actuator disc

P' =

C_D
$C_P = P/$
$C_Q = Q$
$C_T = T/$

$P = Q \cdot \Omega$

$Cp \cdot$

assuming T ≈W in

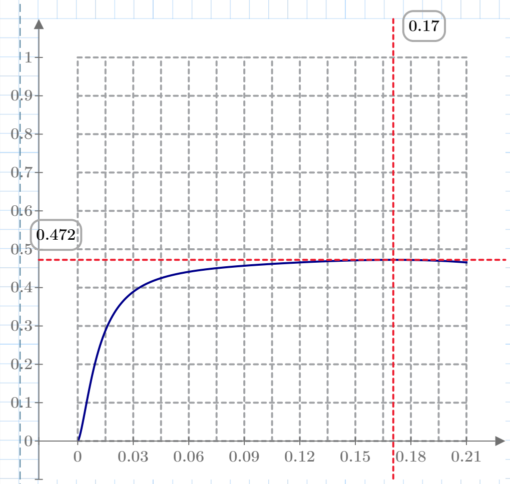
thorugh the disc,

$P_i = Tv$

$P_i = \sqrt{2}$

$G := Grid$

The **Figure of Merit** for a rotor is defined as $FM\left(\theta_c\right) := \dfrac{Ct\left(\theta_c\right)^{\frac{3}{2}}}{\sqrt{2} \cdot Cp\left(\theta_c\right)}$. If we plot that against the thrust coefficient *normalized by the solidity*, we get a standard graph of rotor performance: $\theta_{c\_max} := \textbf{root}\left(\dfrac{d}{dq}FM(q), q, 0, \dfrac{\pi}{6}\right)$



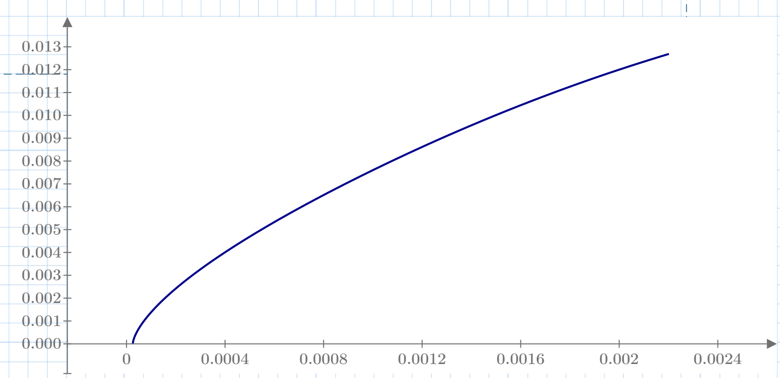
FM

Ct/σ

Typical helicopter performance (of real helicopters) produce Figures of Merit that peak in the vicinity of 0.7, significant effort is expended to improve that number even small amounts.

$\dfrac{Ct\left(\theta_{c\_max}\right)}{\sigma} = 0.17$

$\theta_{c\_max} = 16.012\ deg$



$Ct(qc)$

$Cp(qc)$

Blade Design: The main rotor blade design of a helicopter is a challenge. The choice of airfoils, twist distribution, and planform determine the ability of the helicopter to pereform its mission. Our rotor design above was simplistic--a benign airfoil choice not turning terribly fast. It resulted in a peak Figure of Merit of $FM\left(\theta_{c\_max}\right) = 0.472$, far below the performance of helicopters flying today. Still, an example of the steps to design this blade will prove instructive.

At peak FM, our helicopter will be generating $Thrust\left(\theta_{c\_max}\right) = 4053\ lbf$ of vertical thrust and using $\omega \cdot Torq\left(\theta_{c\_max}\right) + \sqrt{\dfrac{Thrust\left(\theta_{c\_max}\right)^3}{2 \cdot \rho \cdot \pi \cdot R_{tip}^{\ 2}}} = 542\ hp$ to maintain it. Downwash will be $v_i\left(\theta_{c\_max}\right) = 34.57\ \dfrac{ft}{sec}$