Mathcad Community Challenge March 2024 - Perimeter of an Ellipse (using Prime 8 Express)

1. Derive, depict, or show one (or more) of the various approximation formulas / methods for the perimeter of an ellipse.

1a. Introduction

A relatively well known approximation to the perimeter of an ellipse (to the extent that any such approximation is well known!) is π.(a + b), where a and b are the semi-major and semi-minor axis lengths respectively. A number of other approximations are given on the website, https://www.mathsisfun.com/geometry/ellipse-perimeter.html.

However, I see no fun in simply reproducing those formulae here, so I will attempt to derive my own approximations below (of course, it's quite possible that they already exist - there's nothing new under the sun!).

The equation of an ellipse may be written in Cartesian coordinates as:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

which we will rearrange as:

$$y(x, a, b) := b \cdot \sqrt{1 - \left(\frac{x}{a}\right)^2} \qquad ...(1)$$

(we'll take a >= b throughout). Note that the length of the perimeter is independent of its position and orientation, so I've taken its centre to be at coordinates (0, 0), with the semi-major axis lying along the x-axis.

We will only be interested in the positive root for y, as the four-fold symmetry of the ellipse means we only need consider the first quadrant in detail (we'll simply multiply the result by 4 to get the value for the complete ellipse).

We'll need the rate of change of y with respect to x, dy/dx, which is simply determined to be:

$$dydx(x, a, b) := -\frac{b}{a} \cdot \frac{x}{a} \cdot \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \qquad ...(2)$$

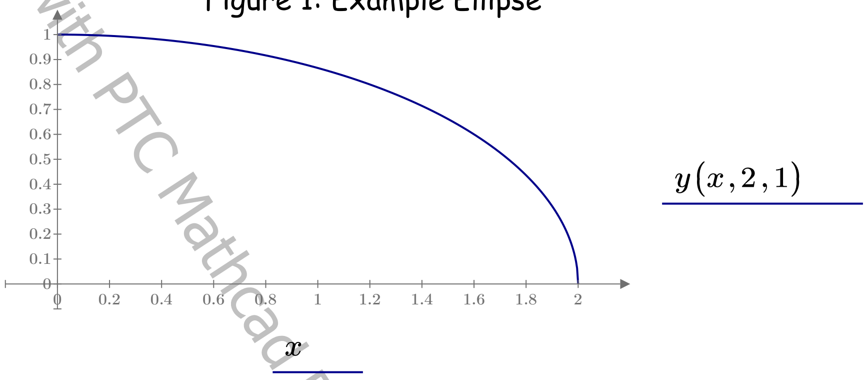
From this we can calculate the "exact" value of the perimeter, S, from:

$$S(a, b) := 4 \cdot \int_0^a \sqrt{1 + dydx(x, a, b)^2}\, dx \qquad ...(3)$$

Of course, this isn't really exact, as it's limited by the numerical precision of the in-built calculational routines! Never-the-less, we'll use it to benchmark our approximations.

Here is a pictorial example of the system of interest:

Figure 1: Example Ellipse



1b. Approximation

Let's start by doing a series expansion of the right-hand side of equation (2):

$$dydx(x,a,b) = -\frac{b}{a}\cdot\frac{x}{a}\cdot\left(1+\frac{1}{2}\cdot\left(\frac{x}{a}\right)^2+\frac{3}{8}\cdot\left(\frac{x}{a}\right)^4\right)$$

where we are ignoring higher order terms.

The kernel of the integral in equation (3) can now be written as:

$$\sqrt{1+\left(\frac{b}{a}\right)^2\cdot\left(\frac{x}{a}\right)^2\cdot\left(1+\frac{1}{2}\cdot\left(\frac{x}{a}\right)^2+\frac{3}{8}\cdot\left(\frac{x}{a}\right)^4\right)^2}$$

on which we do another series expansion to get:

$$\sqrt{1+dydx(x,a,b)^2} = 1+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(\frac{x}{a}\right)^2+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{b}{a}\right)^2\right)\cdot\left(\frac{x}{a}\right)^4 \qquad ...(4)$$

again ignoring higher order terms.

As (4) is a simple polynomial in x, by substituting it into (3) we are able to do the integral very easily. We have (representing the approximate integral by P):

$$P(a,b) = 4\cdot\int_0^a 1+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(\frac{x}{a}\right)^2+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{b}{a}\right)^2\right)\cdot\left(\frac{x}{a}\right)^4\, dx \qquad ...(5)$$

or:

$$P(a,b) := 4 \cdot a \cdot \left(1 + \frac{4}{15} \cdot \left(\frac{b}{a}\right)^2 - \frac{1}{40} \cdot \left(\frac{b}{a}\right)^4\right) \qquad ...(6)$$

1c. Exact vs approximation

Let's examine the relative error of the approximate perimeter, from (6), to that of the "exact" perimeter, from (3). We'll plot the relative errors as a function of the ratio b/a for a couple of values of a.

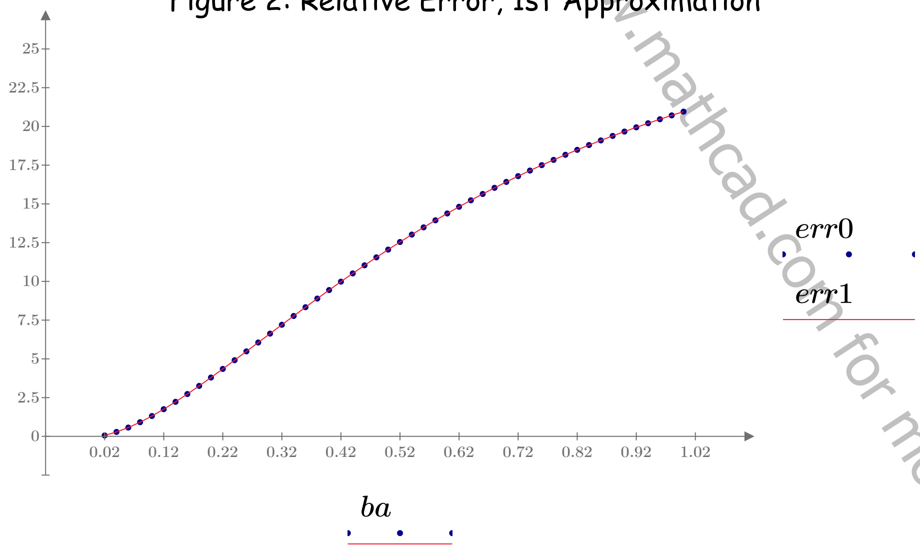
$i := 0..49 \qquad ba_i := \frac{i+1}{50} \qquad a := \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

$b := ba \cdot a^{T}$

Percentage errors:

$$err0 := \left(1 - \frac{\overrightarrow{P\left(a_0, b^{\langle 0 \rangle}\right)}}{\overrightarrow{S\left(a_0, b^{\langle 0 \rangle}\right)}}\right) \cdot 100 \qquad err1 := \left(1 - \frac{\overrightarrow{P\left(a_1, b^{\langle 1 \rangle}\right)}}{\overrightarrow{S\left(a_1, b^{\langle 1 \rangle}\right)}}\right) \cdot 100$$

Figure 2: Relative Error, 1st Approximation



Notice that the relative errors are independent of the magnitude of a. However, clearly, the approximation is unacceptable for a large range of the ratio b/a.

1d. Improved approximation

We see from figure 2 that the approximation is best for small values of the ratio b/a; that is, for the "flatter" ellipses, where the region of steep gradients is a smaller proportion of the curve. This suggests we should try to turn the region of steep gradients into a region of shallow gradients. We can do this by splitting the curve into two parts: the first, where we consider y as a function of x, (as above), and the second, where we consider x as a function of y (think of reflecting the ellipse of figure 1 about the line y = x).

We put the split point where the two gradients (dydx and dxdy) equal each other. Since dxdy = 1/dydx this means we set dydx^2 = 1, or dydx = -1 (we choose the negative root as the gradients are negative in the first quadrant).

So, if the split point is at coordinate (xc, yc), we have (using (2)):

$$-\frac{b}{a}\cdot\frac{xc}{a}\cdot\frac{1}{\sqrt{1-\left(\frac{xc}{a}\right)^2}}=-1$$

from which we obtain: $$xcfn(a,b):=\frac{a}{\sqrt{1+\left(\frac{b}{a}\right)^2}}$$

and (from symmetry): $$ycfn(a,b):=\frac{b}{\sqrt{1+\left(\frac{a}{b}\right)^2}}$$

where I've turned the parameters into functions for later use.

We now replace equation (5) by:

$$Px(xc,a,b)=4\cdot\int_0^{xc}1+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(\frac{x}{a}\right)^2+\frac{1}{2}\cdot\left(\frac{b}{a}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{b}{a}\right)^2\right)\cdot\left(\frac{x}{a}\right)^4\,\mathrm{d}x$$

or:

$$Px(xc,a,b):=4\cdot xc\cdot\left(1+\frac{1}{6}\cdot\left(\frac{b}{a}\right)^2\cdot\left(\frac{xc}{a}\right)^2+\frac{1}{10}\cdot\left(\frac{b}{a}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{b}{a}\right)^2\right)\cdot\left(\frac{xc}{a}\right)^4\right)$$

and have the analogous equations:

$$Py(yc,a,b) = 4 \cdot \int_0^{yc} 1+\frac{1}{2}\cdot\left(\frac{a}{b}\right)^2\cdot\left(\frac{y}{b}\right)^2+\frac{1}{2}\cdot\left(\frac{a}{b}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{a}{b}\right)^2\right)\cdot\left(\frac{y}{b}\right)^4 \, \mathrm{d}y$$

$$Py(yc,a,b) := 4\cdot yc\cdot\left(1+\frac{1}{6}\cdot\left(\frac{a}{b}\right)^2\cdot\left(\frac{yc}{b}\right)^2+\frac{1}{10}\cdot\left(\frac{a}{b}\right)^2\cdot\left(1-\frac{1}{4}\cdot\left(\frac{a}{b}\right)^2\right)\cdot\left(\frac{yc}{b}\right)^4\right)$$

with the total perimeter now given by:

$$P2(xc,yc,a,b) := Px(xc,a,b)+Py(yc,a,b) \qquad ...(7)$$

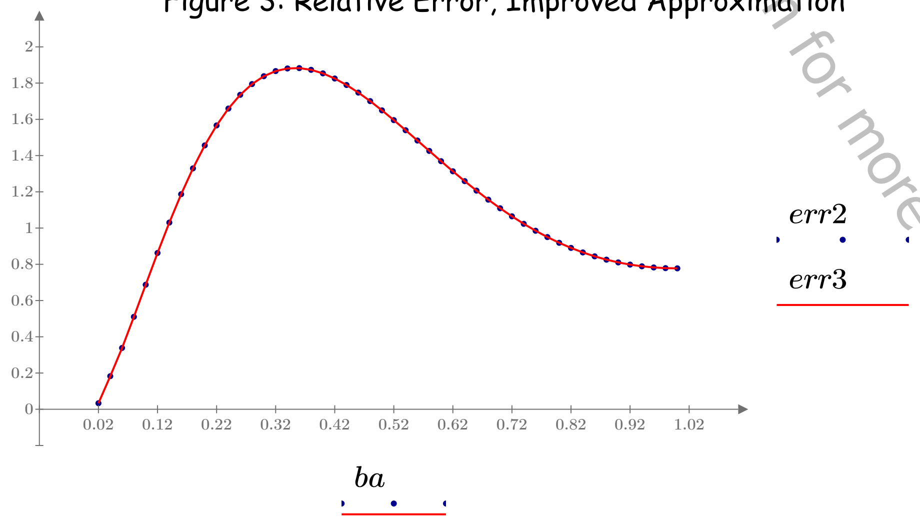
(We could, of course, run the two expressions for Px and Py together to make a single, if rather unwieldy, explicit expression. We could also replace xc and yc where they occur in Px and Py with the values of a and b as they occur in xcfn and ycfn, for an even more unwieldy expression! However, we'll stick with the form of equation (7) here.)

Now let's see how the relative errors vary with the b/a ratio.

$$xc_i := xcfn\left(a_0, b_{i,0}\right) \qquad yc_i := ycfn\left(a_0, b_{i,0}\right) \qquad err2 := \left(1-\frac{\overrightarrow{P2\left(xc, yc, a_0, b^{\langle 0\rangle}\right)}}{\overrightarrow{S\left(a_0, b^{\langle 0\rangle}\right)}}\right)\cdot 100$$

$$xc_i := xcfn\left(a_1, b_{i,1}\right) \qquad yc_i := ycfn\left(a_1, b_{i,1}\right) \qquad err3 := \left(1-\frac{\overrightarrow{P2\left(xc, yc, a_1, b^{\langle 1\rangle}\right)}}{\overrightarrow{S\left(a_1, b^{\langle 1\rangle}\right)}}\right)\cdot 100$$

Figure 3: Relative Error, Improved Approximation



Clearly, we now have a significantly improved explicit approximation, having reduced the maximum relative error from 21% to less than 2%. I think we'll stop there!

2. Create a calculator whereby someone can change the values of the semi-major and semi-minor axis lengths in order to find the perimeter.

2.1. Ellipse perimeter calculator

Replace the values of a, the semi-major axis, and b, the semi-minor axis, in the following table to find the perimeter of the corresponding ellipse. (Default values of a = 2 and b = 1 are shown).

Input table:

a	b
2	1

Perimeter: $S(a,b)=9.688$

3. Create a 3D plot of the perimeter as a function of the ellipse semi-major and semi-minor axis lengths.

3D plotting is not available in the Express version.